

System theory: learning orthogonal multi-wavelets

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Abstract

In this paper we provide a complete and unifying characterization of compactly supported univariate scalar orthogonal wavelets and vector-valued or matrix-valued orthogonal multi-wavelets. This characterization is based on classical results from system theory and basic linear algebra. In particular, we show that the corresponding wavelet and multi-wavelet masks are identified with a transfer function

$$F(z) = A + Bz(I - Dz)^{-1}C, \quad z \in \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\},$$

of a conservative linear system. The complex matrices A , B , C , D define a block circulant unitary matrix. Our results show that there are no intrinsic differences between the elegant wavelet construction by Daubechies or any other construction of vector-valued or matrix-valued multi-wavelets. The structure of the unitary matrix defined by A , B , C , D allows us to parametrize in a systematic way all classes of possible wavelet and multi-wavelet masks together with the masks of the corresponding refinable functions.

Keywords: quadrature mirror filter condition, Unitary Extension Principle, transfer function.

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1 Introduction and notation

Classical results of system theory and basic linear algebra allow us to show that there are no intrinsic differences between the elegant construction of wavelets by Daubechies, in the scalar case, or any other construction of vector or matrix-valued multi-wavelets. We are able to parametrize in a systematic way all possible masks of orthogonal scaling functions and the corresponding wavelet or multi-wavelet masks independently of their dimension. We compare our results with the recent characterization of orthogonal multi-wavelets in [2].

By [29], the Unitary Extension Principle constructions of compactly supported multi-wavelets boil down to manipulations with certain trigonometric polynomials on the unit circle

$$\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}.$$

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Any such construction, requires that the underlying orthogonal scaling function satisfies

$$\phi : \mathbb{R} \rightarrow \mathbb{C}^{K \times M}, \quad \phi = \sum_{\alpha \in \mathbb{Z}} \phi(2 \cdot -\alpha) p_{\alpha}, \quad K, M \in \mathbb{N}, \quad K \leq M, \quad (1)$$

with the mask $\mathbf{p} = \{p_{\alpha} \in \mathbb{C}^{M \times M} : \alpha \in \{0, \dots, n\}\}$, $n \in \mathbb{N}$. The corresponding wavelet or multi-wavelet is defined by

$$\psi : \mathbb{R} \rightarrow \mathbb{C}^{K \times M}, \quad \psi = \sum_{\alpha \in \mathbb{Z}} \phi(2 \cdot -\alpha) q_{\alpha}$$

with the finitely supported mask $\mathbf{q} = \{q_{\alpha} \in \mathbb{C}^{M \times M} : \alpha \in \{0, \dots, n\}\}$, $n \in \mathbb{N}$,

By [9, 18], to ensure the existence of the compactly supported distributional solution of (1), we require that for the symbol

$$p(z) = \sum_{\alpha \in \mathbb{Z}} p_{\alpha} z^{\alpha}, \quad z \in \mathbb{C} \setminus \{0\},$$

there exist $\tilde{M} \leq M$ vectors $\mathbf{v}_1, \dots, \mathbf{v}_{\tilde{M}} \in \mathbb{R}^M$ satisfying

$$p(1)\mathbf{v}_j = \mathbf{v}_j \quad \text{and} \quad p(-1)\mathbf{v}_j = 0, \quad j = 1, \dots, \tilde{M}. \quad (2)$$

Additionally, the other eigenvalues of $p(1)$ should be in the absolute values less than 1. In this case, we say that \mathbf{p} satisfies sum rules of order 1. Sum rules of order 1 imply that the associated multi-wavelet mask possesses a discrete vanishing moment

$$q(1)^* \mathbf{v}_j = 0, \quad j = 1, \dots, \tilde{M}, \quad (3)$$

with the same vectors \mathbf{v}_j are as in (2). In the literature, the cases $\tilde{M} = 1$ and $\tilde{M} = M$ are called the 1-rank and the full rank cases, respectively [9, 10, 26]. The higher smoothness of ϕ imposes additional sum rule conditions on the symbol $p(z)$, see e.g. [7, 21, 23].

In the scalar or full rank cases (i.e. $\tilde{M} = M$), the sum rules of order $\ell + 1$ are equivalent to the existence of the factors $(1 + z)^{\ell}$ in $p(z)$. The vanishing moment conditions of order $\ell + 1$ in these cases guarantee the existence of the factors $(1 - z)^{\ell}$ in $q(z)$.

In the scalar case ($K = M = 1$), the elegant wavelet construction by Daubechies [17] amounts to defining the wavelet mask \mathbf{q} by

$$q_{\alpha} = (-1)^{\alpha} p_{n-\alpha}, \quad \alpha \in \{0, \dots, n\}. \quad (4)$$

In the case $M > 1$, due to the non-commutativity of the matrices p_{α} and q_{α} , the trick in (4) does not apply. Nevertheless, the interest in constructing multi-wavelets has not decreased for the last 30 years and it is motivated, for example, by the fact that the growth of the support of ϕ in this case is decoupled from the smoothness of ϕ and symmetry does not conflict with orthogonality [9].

The constructions of the corresponding matrix-valued masks \mathbf{p} and \mathbf{q} are based on the so-called QMF (quadrature mirror filter) and UEP (Unitary Extension Principle) conditions. To state them, we define the matrix polynomial map

$$F : \mathbb{C} \rightarrow \mathbb{C}^{2M \times 2M}, \quad F(\xi) = \sum_{j=0}^N F_j \xi^j, \quad \xi = z^2, \quad N = \lfloor \frac{n}{2} \rfloor, \quad (5)$$

with the matrix coefficients $F_j \in \mathbb{C}^{2M \times 2M}$, $M \in \mathbb{N}$,

$$F_j = \begin{pmatrix} p_{2j} & q_{2j} \\ p_{2j+1} & q_{2j+1} \end{pmatrix}, \quad j = 0, \dots, N.$$

The entries in the first column of F are usually called the polyphase components of $p(z)$. The QMF-condition states that

$$I - F^*(\xi)F(\xi) = 0, \quad \xi \in \mathbb{T}, \quad (6)$$

and, equivalently, the UEP-conditions are

$$I - F(\xi)F^*(\xi) = 0, \quad \xi \in \mathbb{T}. \quad (7)$$

To use classical results from the theory of linear systems, we look at F in (5) as a holomorphic function on the unit disk

$$\mathbb{D} = \{\xi \in \mathbb{C} : |\xi| < 1\}.$$

The QMF-condition (6) and the maximum principle imply that $F(\xi)$ is contractive for any $\xi \in \mathbb{D}$. Such holomorphic functions, by the von Neumann inequality [31] and by Agler [1], belong to the Schur-Agler class, i.e. are of the form

$$F(\xi) = A + B\xi(I - D\xi)^{-1}C, \quad \xi \in \mathbb{D}, \quad (8)$$

with the unitary matrix

$$ABCD = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{matrix} \mathbb{C}^{2M} \\ \oplus \\ \mathbb{C}^{2MN} \end{matrix} \rightarrow \begin{matrix} \mathbb{C}^{2M} \\ \oplus \\ \mathbb{C}^{2MN} \end{matrix}. \quad (9)$$

Note that the identity in (8) can be equivalently written as

$$\begin{aligned} F(\xi) &= A + \xi B \ell_N(\xi), \\ \ell_N(\xi) &= C + \xi D \ell_N(\xi), \quad \xi \in \mathbb{D}. \end{aligned} \quad (10)$$

This system of equations plays an important role in constructions of appropriate blocks of $ABCD$ matrix.

The paper is organized as follows: In subsection 2.1 we discuss the structure of ℓ_N appearing in the transfer function system (10). In subsection 2.2, we provide the explicit form of the $ABCD$ matrix under the assumptions that F satisfies QMF-conditions on \mathbb{T} , see Theorem 2.5. The compact support of the constructed multi-wavelets is ensured by the property $B D^N = 0$, see Proposition 2.6. The constructions of several compactly supported scaling functions and multi-wavelets are given in Section 4. In subsection 3, we compare our results with the characterization in [2]. The characterization in [2] makes use of the so-called Potapov-Blaschke-products and is also valid for rational F .

We remark that although we prove our results for the case of dilation 2, they all have a straightforward generalization to the case of a general dilation factor.

2 Characterization of orthogonal univariate multi-wavelets

The main goal of this section is to provide the explicit form of the $ABCD$ matrix in (9) for all F that satisfy the QMF-condition (6) or, equivalently, the UEP-condition (7). We start by deriving the structure of ℓ_N in (10). Then determine the explicit structure of the $ABCD$ matrix, see Definition 2.3 and Theorem 2.5. Further properties of the matrix $ABCD$ are studies in subsection 2.3.

2.1 Structure of ℓ_N

To derive the structure of ℓ_N , see Theorem 2.2, we make use of the following straightforward observation.

Proposition 2.1. *The QMF-condition (6) is equivalent to the identity*

$$I - \left(\sum_{k=-N}^{-1} \left(\sum_{\substack{i,j \in \{0, \dots, N\} \\ -i+j=k}} F_i^* F_j \right) \bar{\eta}^k + \sum_{k=0}^N \left(\sum_{\substack{i,j \in \{0, \dots, N\} \\ -i+j=k}} F_i^* F_j \right) \xi^k \right) = 0 \quad (11)$$

for $\xi, \eta \in \mathbb{D}$.

Proof. Note that the QMF-condition (6) is equivalent to

$$I - \sum_{k=-N}^N \left(\sum_{\substack{i,j \in \{0, \dots, N\} \\ -i+j=k}} F_i^* F_j \right) \xi^k = 0, \quad \xi \in \mathbb{T}, \quad (12)$$

i.e. all the coefficients of the above Laurent-polynomial are equal to zero. This implies (11) for all $\xi, \eta \in \mathbb{D}$. Vice versa, if (11) is satisfied, then setting $\bar{\eta} = \eta^{-1}$, we obtain (12). \square

The following result is an important step for determining the structure of the $ABCD$ in (9).

Theorem 2.2. *The polynomial map F satisfies the QMF-condition (6) if and only if*

$$I - F^*(\eta)F(\xi) = (1 - \xi\bar{\eta})\ell_N^*(\eta)\ell_N(\xi), \quad \xi, \eta \in \mathbb{D}, \quad (13)$$

with

$$\ell_N(\xi) = \begin{pmatrix} F_1 & F_2 & \dots & F_{N-1} & F_N \\ F_2 & F_3 & \dots & F_N & 0 \\ \vdots & & & & \vdots \\ F_N & 0 & \dots & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ \xi \\ \vdots \\ \xi^{N-1} \end{pmatrix}. \quad (14)$$

Proof. The proof of " \implies " is by induction on N . Let $\xi, \eta \in \mathbb{D}$. For $N = 1$, due to (5), we have

$$\begin{aligned} I - F^*(\eta)F(\xi) &= I - F_0^*F_0 - F_0^*F_1\xi - F_1^*F_0\bar{\eta} - F_1^*F_1\bar{\eta}\xi \\ &= I - (F_0^*F_0 + F_1^*F_1) - F_0^*F_1\xi - F_1^*F_0\bar{\eta} + (1 - \bar{\eta}\xi)F_1^*F_1. \end{aligned}$$

The QMF-condition (6), by Proposition 2.1, imply

$$I - (F_0^*F_0 + F_1^*F_1) - F_0^*F_1\xi - F_1^*F_0\bar{\eta} = 0,$$

thus, yielding

$$I - F^*(\eta)F(\xi) = (1 - \bar{\eta}\xi)\ell_1^*(\eta)\ell_1(\xi) \quad \text{with} \quad \ell_1(\xi) = F_1.$$

For general N , we first write

$$\begin{aligned} I - F^*(\eta)F(\xi) &= I - \sum_{i=0}^{N-1} F_i^* \bar{\eta}^i \sum_{j=0}^{N-1} F_j \xi^j - F_N^* \bar{\eta}^N \sum_{j=0}^{N-1} F_j \xi^j - \sum_{i=0}^{N-1} F_i^* \bar{\eta}^j F_N \xi^N \\ &\quad - F_N^* F_N \bar{\eta}^N \xi^N. \end{aligned}$$

By induction assumption and Proposition 2.1, we have

$$\begin{aligned} I - \sum_{i=0}^{N-1} F_i^* \bar{\eta}^i \sum_{j=0}^{N-1} F_j \xi^j &= \\ I - \left(\sum_{k=-N+1}^{-1} \left(\sum_{\substack{i,j \in \{0, \dots, N-1\} \\ -i+j=k}} F_i^* F_j \right) \bar{\eta}^k + \sum_{k=0}^{N-1} \left(\sum_{\substack{i,j \in \{0, \dots, N-1\} \\ -i+j=k}} F_i^* F_j \right) \xi^k \right) \\ &\quad + (1 - \bar{\eta}\xi)\ell_{N-1}^*(\eta)\ell_{N-1}(\xi). \end{aligned}$$

To be able to apply the QMF-condition, we write

$$\begin{aligned} &-F_N^* \bar{\eta}^N \sum_{j=0}^{N-1} F_j \xi^j - \sum_{i=0}^{N-1} F_i^* \bar{\eta}^j F_N \xi^N - F_N^* F_N \bar{\eta}^N \xi^N \\ &= - \sum_{j=0}^{N-1} (F_N^* F_j \bar{\eta}^N \xi^j + F_j^* F_N \bar{\eta}^j \xi^N) - F_N^* F_N \bar{\eta}^N \xi^N \\ &= \sum_{j=1}^{N-1} (1 - \bar{\eta}^j \xi^j) (F_N^* F_j \bar{\eta}^{N-j} + F_j^* F_N \xi^{N-j}) - \sum_{j=0}^{N-1} (F_N^* F_j \bar{\eta}^{N-j} + F_j^* F_N \xi^{N-j}) + \\ &\quad (1 - \bar{\eta}^N \xi^N) F_N^* F_N - F_N^* F_N. \end{aligned}$$

Then we get

$$\begin{aligned} I - F^*(\eta)F(\xi) &= I - \left(\sum_{k=-N}^{-1} \left(\sum_{\substack{i,j \in \{0, \dots, N\} \\ -i+j=k}} F_i^* F_j \right) \bar{\eta}^k + \sum_{k=0}^N \left(\sum_{\substack{i,j \in \{0, \dots, N\} \\ -i+j=k}} F_i^* F_j \right) \xi^k \right) \\ &\quad + (1 - \bar{\eta}\xi)\ell_{N-1}^*(\eta)\ell_{N-1}(\xi) \\ &\quad + \sum_{j=1}^{N-1} (1 - \bar{\eta}^j \xi^j) (F_N^* F_j \bar{\eta}^{N-j} + F_j^* F_N \xi^{N-j}) + (1 - \bar{\eta}^N \xi^N) F_N^* F_N. \end{aligned}$$

The QMF-condition and the binomial formula for $(1 - \bar{\eta}^j \xi^j)$ lead to

$$\begin{aligned} I - F^*(\eta)F(\xi) &= (1 - \bar{\eta}\xi) \left(\ell_{N-1}^*(\eta)\ell_{N-1}(\xi) + \right. \\ &\quad \sum_{j=1}^{N-1} \left(\sum_{k=0}^{j-1} (\bar{\eta}\xi)^{j-1-k} \right) (F_N^* F_j \bar{\eta}^{N-j} + F_j^* F_N \xi^{N-j}) + \\ &\quad \left. \sum_{k=0}^{N-1} (\bar{\eta}\xi)^{N-1-k} F_N^* F_N \right). \end{aligned}$$

By the definition of ℓ_{N-1} , we have

$$\ell_{N-1}^*(\eta)\ell_{N-1}(\xi) = \sum_{i=1}^{N-1} (1 \ \dots \ \bar{\eta}^{N-1-i}) \begin{pmatrix} F_i^* \\ \vdots \\ F_{N-1}^* \end{pmatrix} (F_i \ \dots \ F_{N-1}) \begin{pmatrix} 1 \\ \vdots \\ \xi^{N-1-i} \end{pmatrix}.$$

Note that, for $i = 1, \dots, N-1$,

$$\begin{aligned} & (1 \ \dots \ \bar{\eta}^{N-i}) \begin{pmatrix} F_i^* \\ \vdots \\ F_N^* \end{pmatrix} (F_i \ \dots \ F_N) \begin{pmatrix} 1 \\ \vdots \\ \xi^{N-i} \end{pmatrix} = \\ & (1 \ \dots \ \bar{\eta}^{N-i}) \begin{pmatrix} F_i^* \\ \vdots \\ F_{N-1}^* \\ 0 \end{pmatrix} (F_i \ \dots \ F_{N-1} \ 0) \begin{pmatrix} 1 \\ \vdots \\ \xi^{N-i} \end{pmatrix} + \\ & \sum_{j=i}^{N-1} (\bar{\eta}\xi)^{j-i} (F_N^* F_j \bar{\eta}^{N-j} + F_j^* F_N \xi^{N-j}) + (\bar{\eta}\xi)^{N-i} F_N^* F_N \end{aligned}$$

and the reordering of the summands leads to

$$\begin{aligned} & \sum_{j=1}^{N-1} \left(\sum_{k=0}^{j-1} (\bar{\eta}\xi)^{j-1-k} \right) (F_N^* F_j \bar{\eta}^{N-j} + F_j^* F_N \xi^{N-j}) = \\ & \sum_{i=1}^{N-1} \left(\sum_{j=i}^{N-1} (\bar{\eta}\xi)^{j-i} \right) (F_N^* F_j \bar{\eta}^{N-j} + F_j^* F_N \xi^{N-j}). \end{aligned}$$

Therefore, we obtain (13) with ℓ_N in (14). The proof of " \Leftarrow " follows by substituting $\bar{\eta} = \eta^{-1}$ in (13). \square

2.2 Structure of $ABCD$

The main result of this section characterizes all orthogonal wavelets and multi-wavelets in terms of transfer function representations for the analytic map $F : \mathbb{D} \rightarrow \mathbb{C}^{2M \times 2M}$ defined in (5). Such representations involve certain complex matrices, which we define next.

Definition 2.3. For $F_j \in \mathbb{C}^{2M \times 2M}$, $j = 0, \dots, N$, in (5), define the $2M(N+1) \times 2M(N+1)$ block matrix

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) = \left(\begin{array}{c|cccc} F_0 & F_N & \dots & \dots & F_1 \\ \hline F_1 & F_0 & F_N & & \vdots \\ \vdots & F_1 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & F_N \\ F_N & F_{N-1} & \dots & F_1 & F_0 \end{array} \right) \left(\begin{array}{c|c} I & 0 \\ \hline 0 & U \end{array} \right),$$

where the $2MN \times 2MN$ matrix U is given by

$$U = \begin{pmatrix} F_0^* + F_N^* & F_1^* & \dots & F_{N-1}^* \\ F_{N-1}^* & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & F_1^* \\ F_1^* & \dots & F_{N-1}^* & F_0^* + F_N^* \end{pmatrix}.$$

The proof of our main result, Theorem 2.5, relies on the following property of the matrix U in Definition 2.3.

Proposition 2.4. *If the polynomial map F in (5) satisfies the QMF-conditions (6), then the matrix U from Definition 2.3 is unitary.*

Proof. Define

$$P = \begin{pmatrix} 0 & \dots & 0 & I \\ I & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & I & 0 \end{pmatrix} \quad \text{with the identity matrix } I \in \mathbb{R}^{M \times M}.$$

Note that, similarly to the standard definition of circulant matrices,

$$U = (F_0^* + F_N^*)P^0 + \sum_{j=1}^{N-1} F_{N-j}^* P^j.$$

The claim follows by straightforward computations. \square

We are finally ready to state the following characterization of all compactly supported orthogonal wavelet and multi-wavelet masks.

Theorem 2.5. *Let F be a polynomial map in (5). The map F satisfies the QMF-condition (6) if and only if F satisfies*

$$F(\xi) = A + B\xi(I - D\xi)^{-1}C, \quad \xi \in \mathbb{D}, \quad (15)$$

with the unitary map

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{matrix} \mathbb{C}^{2M} \\ \oplus \\ \mathbb{C}^{2MN} \end{matrix} \rightarrow \begin{matrix} \mathbb{C}^{2M} \\ \oplus \\ \mathbb{C}^{2MN} \end{matrix} \quad (16)$$

given in Definition 2.3.

Proof. The proof of " \implies " consists of two parts. Firstly, note that the special choice of the matrices A, B, C, D , Proposition 2.4 and the hypothesis imply that the matrix in (16) is indeed unitary. Next, we show that F satisfies (15). Let $\xi \in \mathbb{D}$. By [6], the identity in (15) is equivalent to the system of equations

$$\begin{aligned} A + \xi B \ell_N(\xi) &= F(\xi) \\ C + \xi D \ell_N(\xi) &= \ell_N(\xi), \end{aligned} \quad (17)$$

with ℓ_N as in Theorem 2.2. Dy the definitions of the matrices A, C and the polynomial map ℓ_N , the system in (17) is equivalent to

$$\begin{aligned} \xi B \ell_N(\xi) &= F_1 \xi + \dots F_N \xi^N = \begin{pmatrix} F_1 & \dots & F_N \end{pmatrix} \begin{pmatrix} \xi \\ \vdots \\ \xi^N \end{pmatrix} \\ \xi D \ell_N(\xi) &= \ell_N(\xi) - C = \begin{pmatrix} F_2 & \dots & F_N & 0 \\ \vdots & & & \vdots \\ F_N & & & \\ 0 & \dots & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi \\ \vdots \\ \xi^N \end{pmatrix}. \end{aligned} \quad (18)$$

After the division of both sides of (18) by ξ and by $U^*U = I$, we get another equivalent system

$$\begin{aligned} & \left(B - \begin{pmatrix} F_0 + F_N & F_{N-1} & \dots & F_1 \end{pmatrix} U \right) \ell_N(\xi) = 0 \\ & \left(D - \begin{pmatrix} F_1 & F_0 + F_N & \dots & F_2 \\ \vdots & \vdots & \ddots & \vdots \\ F_{N-1} & F_{N-2} & \dots & F_0 + F_N \\ 0 & 0 & \dots & 0 \end{pmatrix} U \right) \ell_N(\xi) = 0. \end{aligned} \quad (19)$$

Observe that the QMF-conditions yield

$$\begin{aligned} U \ell_N(\xi) &= \left(\begin{pmatrix} F_N^* & \dots & 0 \\ \vdots & \ddots & \vdots \\ F_1^* & \dots & F_N^* \end{pmatrix} + \begin{pmatrix} F_0^* & \dots & F_{N-1}^* \\ \vdots & \ddots & \vdots \\ 0 & \dots & F_0^* \end{pmatrix} \right) \ell_N(\xi) \\ &= \begin{pmatrix} F_N^* & \dots & 0 \\ \vdots & \ddots & \vdots \\ F_1^* & \dots & F_N^* \end{pmatrix} \ell_N(\xi) \end{aligned}$$

Thus, the QMF- and UEP-conditions imply that the matrices B and D in Definition 2.3 satisfy (19). Indeed,

$$\begin{aligned} - \begin{pmatrix} F_0 & 0 & \dots & 0 \end{pmatrix} U \ell_N(\xi) &= \\ &= - \begin{pmatrix} F_0 & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} F_N^* & \dots & 0 \\ \vdots & \ddots & \vdots \\ F_1^* & \dots & F_N^* \end{pmatrix} \ell_N(\xi) = 0 \end{aligned}$$

and

$$\begin{aligned} & \left(D - \begin{pmatrix} F_1 & F_0 + F_N & \dots & F_2 \\ \vdots & \vdots & \ddots & \vdots \\ F_{N-1} & F_{N-2} & \dots & F_0 + F_N \\ 0 & 0 & \dots & 0 \end{pmatrix} U \right) \ell_N(\xi) = \\ & \left(\begin{pmatrix} F_0 & 0 & \dots & 0 \\ F_1 & F_0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ F_{N-1} & F_{N-2} & \dots & F_0 \end{pmatrix} - \begin{pmatrix} F_1 & F_0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ F_{N-1} & F_{N-2} & \dots & F_0 \\ 0 & 0 & \dots & 0 \end{pmatrix} \right) U \ell_N(\xi) = 0. \end{aligned}$$

The proof of " \Leftarrow " follows by a standard argument from system theory. By [1], the identity (15) is equivalent to (13) with ℓ_N in (14). Thus, by Theorem 2.2, F satisfies the QMF-conditions (6). \square

2.3 Further properties of $ABCD$

In this section, we analyze the properties of the matrices B and D that guarantee that the representation in (15) leads to a polynomial F .

Proposition 2.6. *If F of degree N in (5) satisfies QMF-condition (6), then the matrices B and D from Definition 2.3 satisfy*

$$B D^N = 0.$$

Proof. Write $B = \tilde{B}U$ and $D = \tilde{D}U$ with U from Definition 2.3. Note that, due to the invertibility of U , we only need to show that $\tilde{B}(U\tilde{D})^N = 0$. Thus, to prove the claim, we show that, for $\ell = N-1, \dots, 0$,

$$(0_{N-\ell-1} \ F_N \ \dots \ F_{N-\ell})U\tilde{D} = (0_{N-\ell} \ F_N \ \dots \ F_{N-\ell+1}). \quad (20)$$

Then, we get

$$\text{For } \ell = N-1: \quad \tilde{B}U\tilde{D} = (F_N \ \dots \ F_2 \ F_1)U\tilde{D} = (0 \ F_N \ \dots \ F_2),$$

$$\text{For } \ell = N-2: \quad \tilde{B} \left(U\tilde{D} \right)^2 = (0 \ F_N \ \dots \ F_2)U\tilde{D} = (0 \ 0 \ F_N \ \dots \ F_3)$$

and so on. For $\ell = 0$, we get a zero row vector on the right-hand side.

Let $\ell \in \{0, \dots, N-1\}$. First, note that, due to the structure of U and \tilde{D} ,

$$\begin{aligned} U &= U_1 + U_2 := \begin{pmatrix} F_0^* & F_1^* & \dots & F_{N-1}^* \\ \vdots & \ddots & \ddots & \vdots \\ 0 & & \ddots & F_1^* \\ 0 & 0 & \dots & F_0^* \end{pmatrix} + \begin{pmatrix} F_N^* & \dots & 0 & 0 \\ F_{N-1}^* & \ddots & & 0 \\ \vdots & \ddots & \ddots & \vdots \\ F_1^* & \dots & F_{N-1}^* & F_N^* \end{pmatrix}, \\ \tilde{D} &= \tilde{D}_1 + \tilde{D}_2 := \begin{pmatrix} F_0 & \dots & 0 & 0 \\ F_1 & \ddots & & 0 \\ \vdots & \ddots & \ddots & \vdots \\ F_{N-1} & \dots & F_1 & F_0 \end{pmatrix} + \begin{pmatrix} 0 & F_N & \dots & F_2 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & & \ddots & F_N \\ 0 & 0 & \dots & 0 \end{pmatrix}. \end{aligned}$$

By QMF, $U_2\tilde{D}_1 = 0$ and, by UEP, $(0_{N-\ell-1} \ F_N \ \dots \ F_{N-\ell})U_1 = 0$. Thus,

$$\begin{aligned} (0_{N-\ell-1} \ F_N \ \dots \ F_{N-\ell})U\tilde{D} &= (0_{N-\ell-1} \ F_N \ \dots \ F_{N-\ell})U_2\tilde{D}_2 \\ &= \left(\sum_{\substack{-i+j=\ell-N+k \\ i \in \{N-\ell, \dots, N\}}} F_i F_j^* \right)_{k=1}^N \tilde{D}_2. \end{aligned}$$

The UEP implies

$$\sum_{\substack{-i+j=\ell-N+k \\ i \in \{N-\ell, \dots, N\}}} F_i F_j^* = \begin{cases} I - \sum_{\substack{-i+j=0 \\ j \in \{0, \dots, k-1\}}} F_i F_j^*, & \text{if } k = N-\ell, \\ - \sum_{\substack{-i+j=\ell-N+k \\ j \in \{0, \dots, k-1\}}} F_i F_j^*, & \text{otherwise,} \end{cases} \quad k = 1, \dots, N.$$

Therefore,

$$\left(\sum_{\substack{-i+j=\ell-N+k \\ i \in \{N-\ell, \dots, N\}}} F_i F_j^* \right)_{k=1}^N = (0_{N-\ell-1} \ I \ 0_\ell) - \sum_{\substack{\ell-K+k \leq 0 \\ k \in \{1, \dots, N\}}} F_{N-k-\ell} (0_{k-1} \ F_0^* \ \dots \ F_{N-k}^*).$$

Multiplication by \tilde{D}_2 of both sides of the above equation, by the QMF-condition, yields the claim. \square

3 Special case $N = 1$

In this section, we consider the special situation of polynomials F of degree $N = 1$. The following Lemma 3.1 is crucial for comparison of Theorem 2.5 with [2, Theorem 3.1] and also for our specific constructions in section 4.

Lemma 3.1. *Let A, B, C, D be matrices in $\mathbb{R}^{2M \times 2M}$. The following two sets of conditions (I) and (II) are equivalent.*

$$(I) \left\{ \begin{array}{ll} \text{The block matrix } \begin{pmatrix} A & B \\ C & D \end{pmatrix} \text{ is unitary,} & (I.a) \\ DC = 0, \quad BC = C, & (I.b) \\ B = CU, \quad D = AU, \quad U = A^* + C^*. & (I.c) \end{array} \right.$$

$$(II) \left\{ \begin{array}{ll} B^2 = B, \quad B^* = B. & (II.a) \\ D = I - B; & (II.b) \\ A = DU^*, \quad C = BU^*, \quad U = A^* + C^*. & (II.c) \end{array} \right.$$

Proof. Note first that the conditions (I.c) and (II.c) are equivalent.

Assume that (I) are satisfied. The unitarity of the $ABCD$ matrix, in particular $A^*A + C^*C = I$ and $A^*B + C^*D = 0$, and (II.c) imply that U is unitary

$$UU^* = (A^* + C^*)(A + C) = A^*A + C^*C + A^*C + C^*A = I + (A^*B + C^*D)U^* = I.$$

Next, $BC = C$ and (II.c) yield

$$B^2 = B B = B(CU) = (BC)U = CU = B.$$

Definitions of A and C in (II.c) imply $AC^* = \frac{1}{2}AC^* + \frac{1}{2}DB^*$. The unitarity of the $ABCD$ matrix, in particular $AC^* + DB^* = 0$ yields

$$B^* = UC^* = (A + C)C^* = AC^* + CC^* = CC^* = B.$$

By (II.c) we obtain $U^* = A + C = (D + B)U^*$. Thus, $U^*U = I$, leads to $D = I - B$.

Assume that (II) are satisfied. By (II.a) and (II.b), the matrix D also satisfies $D^2 = D$ and $D^* = D$. Thus, by the definitions of A and C in (II.c), $U = A^* + C^*$ is unitary

$$U^*U = D^2 + B^2 + DB + BD = (I - B)^2 + B^2 + (I - B)B + B(I - B) = I.$$

Moreover, (I.a) – (I.c) yield

$$A^*A + C^*C = U(D^*D + B^*B)U^* = U((I - B)^2 + B^2)U^* = UU^* = I,$$

which also proves that $B^*B + D^*D = I$. Furthermore,

$$A^*B + C^*D = UD^*B + UB^*D = U((I - B)B + B(I - B)) = 0.$$

Similarly, $B^*A + D^*C = 0$. Next, we show that $DC = 0$ and $BC = C$. From (II.a) and (II.b) we get

$$DC = (I - B)C = (I - B)BU^* = (B - B^2)U^* = 0$$

and, by the definition of C ,

$$BC = BB^*U^* = B^2U^* = BU^* = C,$$

which concludes the proof. \square

4 Examples

This section we illustrate our results with several examples. In particular, for $N = 1$, the examples illustrate the strength of the algorithm given by the conditions (II) in Lemma 3.1. This algorithm allows us to characterize all possible wavelet and multi-wavelet masks with support on $[0, 2]$ or $[0, 3]$. In subsection 4.2, we show how to apply the result of Theorem 2.5 for construction of F in (5) of degree $N = 2$ with support on $[0, 5]$.

4.1 Wavelets and multi-wavelets supported on $[0, 2]$ or on $[0, 3]$

Several properties of F in (5) are similar in the scalar ($K = M = 1$) and full rank ($K = M > 1$) cases. The corresponding masks are characterized in subsection 4.1.1. The rank one case ($1 = K < M$) is considered in subsection 4.1.2.

4.1.1 Wavelets and full rank multi-wavelets

We first consider the full rank $K = M$ matrix case, which includes the wavelet case $K = M = 1$. Note first that the full rank requirement in the case $K = M$ uniquely determines the unitary matrix U . In fact, since $U = F_0^* + F_1^*$, the first order sum rule/vanishing moments conditions (2)-(3) are equivalent to

$$\begin{pmatrix} I_M & I_M \\ I_M & -I_M \end{pmatrix} U^* = \sqrt{2} \begin{pmatrix} I_M & 0 \\ 0 & W \end{pmatrix}, \quad W^*W = I_M.$$

Therefore,

$$U = \frac{\sqrt{2}}{2} \begin{pmatrix} I_M & W \\ I_M & -W \end{pmatrix}.$$

By Lemma 3.1, the masks \mathbf{p} and \mathbf{q} are, thus, determined by the choice of the projection B . In the scalar case $K = M = 1$, there are only three choices for B : identity or two one-parameter families

$$B_+ = \begin{pmatrix} b & \sqrt{b-b^2} \\ \sqrt{b-b^2} & 1-b \end{pmatrix} \quad \text{and} \quad B_- = \begin{pmatrix} b & -\sqrt{b-b^2} \\ -\sqrt{b-b^2} & 1-b \end{pmatrix}, \quad b \in \mathbb{R}. \quad (21)$$

Once a particular B is chosen, set $F_0 = A = (I - B)U^*$, $F_1 = C = BU^*$. To recover Haar masks \mathbf{p} and \mathbf{q} choose $B = I_2$ or $b = 1$. To impose additional sum rules/vanishing moments [7, 21, 23] for $k = 1$ we solve for b

$$\begin{pmatrix} 0 & -1 & 2 & -3 \\ 0 & 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} F_0 \\ F_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (22)$$

This system yields a unique solution $b = \frac{3}{4}$, determining the Daubechies (D4) masks \mathbf{p} and \mathbf{q} with the supports $[0, 3]$.

In the case $K = M = 2$, there are several choices for the projection B . If we look for the masks \mathbf{p} and \mathbf{q} with supports $[0, 2]$, then the only possible projections B are given by

$$B = \begin{pmatrix} B_{\pm} & 0_2 \\ 0_2 & 0_2 \end{pmatrix}, \quad (23)$$

where the blocks B_{\pm} are given in (21) and 0_2 are 2×2 zero blocks. Note, however, that these choices of B lead to *essentially diagonal* matrix-valued masks \mathbf{p} and \mathbf{q} specified in [11]. Such essentially diagonal matrix-valued masks \mathbf{p} and \mathbf{q} are equivalent to some scalar masks \mathbf{p}_j and \mathbf{q}_j , $j = 1, 2$, since \mathbf{p} and \mathbf{q} are jointly diagonalizable. This means that there are only trivial full rank matrix-valued masks \mathbf{p} and \mathbf{q} supported on $[0, 2]$.

If we consider $K = M = 2$ and look for the masks with supports $[0, 3]$, then we retrieve e.g. all the full rank families of masks \mathbf{p} and \mathbf{q} in [15]. For example, the ones in [15, Table A.4] are obtained for the projections

$$B = \begin{pmatrix} 1 - b_1 & 0 & 0 & \sqrt{b_1 - b_1^2} \\ 0 & 1 - b_2 & \sqrt{b_2 - b_2^2} & 0 \\ 0 & \sqrt{b_2 - b_2^2} & b_2 & 0 \\ \sqrt{b_1 - b_1^2} & 0 & 0 & b_1 \end{pmatrix}, \quad 0 \leq b_1, b_2 \leq 1.$$

Whereas, the masks \mathbf{p} and \mathbf{q} in [15, Table A.3] come from the projection

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{4}(b+1)^2 & \frac{1}{4}(b^2-1) & -\frac{\sqrt{2}}{4}\sqrt{-b^2+1}(b+1) \\ 0 & \frac{1}{4}(b^2-1) & \frac{1}{4}(b-1)^2 & -\frac{\sqrt{2}}{4}\sqrt{-b^2+1}(b-1) \\ 0 & -\frac{\sqrt{2}}{4}\sqrt{-b^2+1}(b+1) & -\frac{\sqrt{2}}{4}\sqrt{-b^2+1}(b-1) & \frac{1}{2}(-b^2+1) \end{pmatrix}, \quad |b| \leq 1.$$

As in the scalar case, the free parameters b , b_1 and b_2 are determined by imposing additional sum rules/vanishing moment conditions. These conditions are similar to the ones in (22), due to the nature of the full rank case:

$$\begin{pmatrix} 0 & -1 & 2 & -3 \\ 0 & 1 & 2 & 3 \end{pmatrix} \text{ replaced by } \begin{pmatrix} 0_2 & -I_2 & 2I_2 & -3I_2 \\ 0_2 & I_2 & 2I_2 & 3I_2 \end{pmatrix}.$$

4.1.2 1-rank orthogonal multi-wavelets

In this subsection we relax the full rank requirement and consider the multi-wavelet (rank 1) setting with $1 = K < M = 2$. If we require the support of the masks \mathbf{p} and \mathbf{q} to be $[0, 2]$, then the projection B is as in (23). To impose sum rule/vanishing moment conditions on the unitary matrix U , we consider, for some non-zero $v = (v_1, v_2)^T \in \mathbb{R}^2$, the system

$$\begin{pmatrix} v_1 & v_2 & 0 & 0 \end{pmatrix} U = \frac{\sqrt{2}}{2} \begin{pmatrix} v_1 & v_2 & v_1 & v_2 \end{pmatrix}. \quad (24)$$

and

$$\begin{pmatrix} v_1 & v_2 & v_1 & v_2 \end{pmatrix} U^* = \sqrt{2} \begin{pmatrix} v_1 & v_2 & 0 & 0 \end{pmatrix}. \quad (25)$$

Note that, by [7, 21], we can restrict our attention w.l.g. to the case $v = (1, 0)^T$. (However, one can allow for different $v \in \mathbb{R}^2$ to be able to reproduce other known constructions.) Since

$v = \hat{\phi}(0)$, this happens for example under the assumption that the components of $\phi = (\phi_1, \phi_2)$ are symmetric/antisymmetric, respectively, around the center of their support, see e.g. [8]. In this case, the first row of U can be determined from

$$\begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} U = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & 0 & 1 & 0 \end{pmatrix}.$$

To impose the symmetry/antisymmetry assumptions, we set the zero entry of the mask \mathbf{p} to be $p_0 = S p_2 S$ and its first entry p_1 to be diagonal. We use $S := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then F_0 and F_1 are diagonal matrices and the matrix U , which depends only on one parameter, is one of the following matrices

$$V_1 \begin{pmatrix} U_1 & U_2 \\ -U_2 & U_1 \end{pmatrix} V_2,$$

where

$$U_1 = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 \\ 0 & \ell \end{pmatrix}, \quad U_2 = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 \\ 0 & \sqrt{1-\ell^2} \end{pmatrix}, \quad \ell \in \mathbb{R},$$

and

$$V_1 \in \left\{ \begin{pmatrix} I & O \\ O & \pm I \end{pmatrix}, \begin{pmatrix} I & O \\ O & \pm S \end{pmatrix} \right\}, \quad V_2 \in \left\{ \begin{pmatrix} I & O \\ O & I \end{pmatrix}, \begin{pmatrix} I & O \\ O & S \end{pmatrix} \right\}.$$

The Chui-Lian multi-wavelets [8] correspond to the choice $\ell = -\frac{\sqrt{14}}{4}$ and $b = \frac{1}{2}$ in (23).

The next example, is related to a special type of multi-wavelet systems proposed in [25]. By similar argument as the ones used in [5, 14], the authors in [25] derive proper pre-filters associated to any multi-wavelet basis. The construction is based on the requirement that the mask \mathbf{p} preserves the constant data which make any pre-filtering step obsolete. Preservation of constants is equivalent to the choice $v = (1, 1)^T$ in (24)-(25). In order to reduce the degrees of freedom, we impose some symmetry constraints on \mathbf{p} and \mathbf{q} (see [25]) directly on the matrix U . Thus, we split U into four symmetric blocks. One of such matrices U (the other possibilities differ only by sign changes) is given by

$$U = \begin{pmatrix} \ell & \frac{1}{2} - \ell & \frac{\sqrt{2}}{4}(1 + J_\ell^-) & \frac{\sqrt{2}}{4}(1 - J_\ell^-) \\ \frac{1}{2} - \ell & \ell & \frac{\sqrt{2}}{4}(1 - J_\ell^-) & \frac{\sqrt{2}}{4}(1 + J_\ell^-) \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{2\sqrt{2}\ell+1}{2(8\ell^2-1)}J_\ell^+J_\ell^- & -\frac{2\sqrt{2}\ell+1}{2(8\ell^2-1)}J_\ell^+J_\ell^- & -\frac{1}{2}J_\ell^+ & \frac{1}{2}J_\ell^+ \end{pmatrix}$$

with $J_\ell^\pm = \sqrt{1 \pm 8\ell^2 \mp 4\sqrt{2}\ell}$. The masks \mathbf{p} and \mathbf{q} in [25] correspond to $\ell = \frac{\sqrt{2}}{8}(2 - \sqrt{7})$ and to the value $b = 1$ in (23).

4.2 Wavelets with support $[0, 5]$.

In this section, we consider the case $K = M = 1$ and $N = 2$ and apply the method for determining the masks of Daubechies (D6) given by Theorem 2.5. By such theorem, the unitary matrix

$$U^* = \begin{pmatrix} F_0 + F_2 & F_1 \\ F_1 & F_0 + F_2 \end{pmatrix}$$

contains already all the information about the unitary $F(z)$ we aim to determine. Imposing the sum rules/vanishing moments of order 3 on U^* leads to

$$\begin{aligned} F_0 &= \sqrt{2} \begin{pmatrix} p_0 & q_0 \\ p_1 & q_1 \end{pmatrix}, \\ F_1 &= \sqrt{2} \begin{pmatrix} \frac{1}{8} - 4p_0 + 2p_1 & \frac{1}{8} - 4q_0 - 2q_1 \\ \frac{3}{8} - 2p_0 & -\frac{3}{8} + 2q_0 \end{pmatrix}, \\ F_2 &= \sqrt{2} \begin{pmatrix} \frac{3}{8} + 3p_0 - 2p_1 & \frac{3}{8} + 3q_0 + 2q_1 \\ \frac{1}{8} + 2p_0 - p_1 & -\frac{1}{8} - 2q_0 - q_1 \end{pmatrix}, \quad p_0, p_1, q_0, q_1 \in \mathbb{R}. \end{aligned}$$

The condition $U^*U = I$ reduces the 4 parameters to one, $t = q_0$. This requires us to solve 4 quadratic equations in 4 unknowns. We obtain four possible solutions depending on t . We present only one of them that corresponds to $D6$. The others are the same up to a sign change.

$$p_0 = \frac{1}{16}(1 + a_t), \quad p_1 = \frac{1}{8}(2 - 8t + a_t), \quad q_1 = -\frac{1}{16}(1 + 8t - a_t),$$

where $a_t = \sqrt{-255t^2 + 32t + 7}$. The parameter t is determined by solving one equation with the radical a_t and yields $t = \frac{1}{32} + \frac{1}{32}\sqrt{10} \mp \frac{1}{32}\sqrt{5 + 2\sqrt{10}}$.

5 Potapov-Blaschke factorizations: scalar case

In this subsection, we consider only the cases $K = M = 1$ and $N = 1, 2$. We think the corresponding examples are sufficient for the comparison of our results with the ones in [2]. The case of $N = 1$ is of special interest as it directly establishes a link between our results and the results in [2].

It has been observed already in [28], see also e.g. [20, Theorem 4.3], that any trigonometric polynomial of degree N , which is unitary on the unit circle, possesses a factorization into so-called Blaschke-Potapov factors. These factorizations were applied for constructions of finite impulse response filter in [2]. In the case $N = 1$, the result of Lemma 3.1 also leads us to Blaschke-Potapov factors. Indeed in this case $A = DU^* = (I - B)U^*$ and $C = BU^*$ and hence,

$$F(\xi) = A + Cz = DU^* + BU^*\xi = (I - B)U^* + BU^*\xi = (I - B + B\xi)U^*, \quad |\xi| = 1.$$

For factorizations of higher degree F into Blaschke-Potapov factors we use the matrices B and U constructed via the algorithm in (II) Lemma 3.1. In general, any unitary $F(\xi) \in \mathbb{C}^{2M \times 2M}$, $|\xi| = 1$, of degree N possesses a factorization

$$F(\xi) = \prod_{j=1}^N (I - B_j + B_j\xi)U_j, \quad U_j^*U_j = I, \quad j = 1, \dots, N,$$

where B_j are some rank-1 projections.

For $N = 2$, the Daubechies (D6) scaling and wavelet masks are obtained by considering

$$F(\xi) = (I - B_1 + B_1\xi)(I - B_2 + B_2\xi)U^*$$

for some B_1 and B_2 in (21). To determine the corresponding parameters b_1 and b_2 , as mentioned above, we determine the corresponding F_j and, then, impose further the sum

rules/vanishing moments of order 2

$$\sum_{j=0}^2 \begin{pmatrix} (2j)^k & -(2j+1)^k \\ (2j)^k & (2j+1)^k \end{pmatrix} F_j = \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}, \quad k = 1, 2,$$

where $*$ symbolizes the matrix entries that do not contribute to our computations. We get

$$b_1 = \frac{5}{4} - \frac{1}{8} \sqrt{10}, \quad b_2 = \frac{1}{8} \sqrt{10},$$

or more explicitly

$$B_1 = \begin{pmatrix} 5/4 - 1/8 \sqrt{10} & -1/8 \sqrt{-30 + 12 \sqrt{10}} \\ -1/8 \sqrt{-30 + 12 \sqrt{10}} & -1/4 + 1/8 \sqrt{10} \end{pmatrix},$$

$$B_2 = \begin{pmatrix} 1/8 \sqrt{10} & -1/8 \sqrt{-10 + 8 \sqrt{10}} \\ -1/8 \sqrt{-10 + 8 \sqrt{10}} & 1 - 1/8 \sqrt{10} \end{pmatrix}.$$

To obtain b_1 and b_2 , we, additionally, need to solve one quadratic and one equation with the radicals. Thus, the computational effort is exactly the same as in section 4.2.

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